Some Effects of Spatial Resolution in the Calculation of Slope Using the Spatial Derivative

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Abstract

Many persons talk about calculating slope and aspect from gridded DEMs as if there were only one way to do so, but the derived values of slope and aspect vary with the algorithms used, the selection and spacing of the elevation points, and the vertical precision of the data. This paper examines the effects of five different data point selection procedures employed in the calculation of slope and aspect in gridded DEMs. In all cases, slope is calculated with the spatial derivative using data from USGS gridded DEMs.

In the case of the 7 1/2 minute DEMs, horizontal spacing will always be a multiple of 30 meters. Elevation values may be reported in meters or in feet, with the latter giving greater precision.

Statistics tabulated from large datasets show that the more extensive the data point selection process, the more smoothed-out are the resulting values of slope. In one case a maximum of 42 degrees was calculated using the finest data point selection procedure, while the maximum slope was only 37 degrees using the most coarse procedure.

Introduction

In concept slope is fairly straight forward in 2D or 3D space. It is defined by the first derivative, which is quite easy to compute if the line or surface is defined mathematically. However, few landform surfaces are defined mathematically although it is becoming more common to fit a function to empirically sampled matrices of elevation.

In practice, determining the direction of slope and measuring the degree of slope is not always that straight forward. It is easy to define slope as 'rise over run,' but rise and run are determined at the discretion of the analyst and not everyone will see the same run or rise in the field. Whether scaling the values from a topographic map or measuring lengths in the field, rise and run have to be given in discrete units. Run has to be oriented perpendicular to the contour and the longer the length of the run, the more likely the run is not everywhere perpendicular to the contour. Rise can only be calculated after run has been decided upon. Unless a surface is truly planar, at best slope will be a generalization of the trend if selected appropriately. Thus, the subjectivity of defining slope.

When working with digital elevation data, the ability to calculate slope values is limited by the format of the elevation values. If the digital data were collected in the field or in a stereo model by selecting significant points and breaks in slope to construct a TIN (triangulated irregular network), then slope was a factor in determining where the elevation values would be collected. In such a case, slope direction and degree are the inherent product of the triangular patches defined by the TIN.

Gridded Arrays of Elevations

In gridded DEMs, the orientation and spacing of the array of elevation values is imposed irrespective of the shape or slope of the land surface and therefore slope has to be derived from the grid of elevations. Because no a priori knowledge is used in selecting the grid spacings, one has to start with the assumption that each elevation is of equal importance. There are many ways slope can be derived from such grids of elevation. Some packages fit a TIN to the gridded surface and in the process eliminate the elevations of lesser significance. Obviously, with the elimination of a large proportion of the elevations there will be a tendency towards generalization. Some persons fit functions to the grid of elevations and derive slope as if the surface were continuous.

Another way to calculate slope from gridded matrices of elevation is to step through the matrix with a roving window looking at the discrete differences of neighbors. There are a number of ways slope has been calculated based on the discrete differences of neighbors (Burrough, 1986, Evans, 1980, Papo and Gelbman, 1984, Ritter, 1986, Snyder, 1983, Struve, 1977). No attempt has been made in this paper to identify all such approaches.

The Spatial Derivative

The most common algorithm for calculating slope from a gridded array of elevations is to compute the 'spatial derivative' (Tobler, 1970, Monmonier, 1982, Burrough, 1986). The discrete version of the spatial derivative is:

$$Slope^{o} = ArcTan\sqrt{\left(\frac{\Delta Z_x}{\Delta X}\right)^2 + \left(\frac{\Delta Z_y}{\Delta Y}\right)^2} \qquad (1)$$

With elevations already in a gridded form Delta X and Delta Y are restricted to multiples of the distance of the grid spacing. Delta Z is constrained by the precision with which the elevations were collected and reported. For the standard USGS 7 1/2 minute DEMs in which the grid is oriented along to the UTM coordinates, Delta X and Y are multiples of 30 meters and Delta Z is reported to the nearest meter. I have encountered a few 7 1/2 minute DEMs where the horizontal spacing is in meters but the vertical values (Delta Z) are reported in feet. In the arc-second DEMs produced or distributed by USGS, Delta X will be equal to the length of Delta Y times the cosine of the latitude at Y. In all of the arc-second DEMs I have seen, Delta Z is reported in meters.

A	B	C
D	E	F
G	H	I

Figure 1: A gridded array of sample points: assume these form a square grid with points spaced 30 meters apart as occurs in the USGS 7 1/2 minute gridded DEMs.

Given the 3 X 3 matrix of elevations in Figure 1 there are a number of ways the spatial derivative can be calculated. The normal way would be to calculate slope for a point at E (call this the 5PT case for four elevations are used and the slope value is applied to the 5th elevation).

$$Slope^{o} \ (5PT \ case) = ArcTan\sqrt{\left(\frac{D-F}{60}\right)^2 + \left(\frac{H-B}{60}\right)^2} \qquad (2)$$

Horn (1981) noted that for certain classes of surfaces numerical analysis teaches us that a weighted average would give a better result and he would substitute for B, (A + B + B + C)/4 and for D, (A + D + D + G)/4, etc. Again, slope is given for E (call this the 9PT case for 9 points are used in the calculation and application of the value).

With B2 = (A + B + B + C) / 4 and D2 = (G + D + D + A) / 4 and
H2 = (G + H + H + I) / 4 and F2 = (C + F + F + I) / 4

$$Slope^{o} \ (9PT \ case) = ArcTan\sqrt{\left(\frac{D2-F2}{60}\right)^2 + \left(\frac{H2-B2}{60}\right)^2} \qquad (3)$$

I became interested in how well the spatial derivative defined the slope in different situations and developed some indices. In working with the spatial derivative I realized that the spatial derivative could be applied to sub-units of the normal grid. The only condition required to compute the spatial derivative is that the axes of X and Y be perpendicular. Thus, it is possible to compute slope for a point at the center of a 30 meter square cell, such as that formed by A, B, D, and E, with the axes of X and Y oriented along the diagonals (call this the 4PT case).

$$Slope^{o} \ (4PT \ case) = ArcTan\sqrt{\left(\frac{D-B}{30\sqrt{2}}\right)^2 + \left(\frac{A-E}{30\sqrt{2}}\right)^2} \qquad (4)$$

Subsequent work with this model revealed that an equivalent result would be realized by using the averages of the side, top and bottom elevations and with the X and Y axes along the normal orientations.

With AD = (A + D) / 2 and DE = (D + E) / 2 and
EB = (E + B) / 2 and AB = (B + A) / 2

$$Slope^{o} \ (4PT \ case) = ArcTan\sqrt{\left(\frac{AD-EB}{30}\right)^2 + \left(\frac{DE-AB}{30}\right)^2} \qquad (5)$$

Following this same logic, the matrix can still be subdivided into smaller units which can be used to compute the spatial derivative. One could assume that the axes forming two sides of a patch give a fairly good definition to the patch, at least near the junction of the axes. On this basis then, slope can be calculated for square patches 15 meters on a side and the value applied to the center of each of the smaller patches (the 3E case, or 3 point case using the edges). Thus, for the example above

$$Slope^{O} \ (3E\ case) = ArcTan\sqrt{\left(\frac{A-B}{30}\right)^2 + \left(\frac{D-A}{30}\right)^2} \tag{6}$$

A similar calculation would be made for the patch defined by (A-B and B-E), (D-E and B-E), and (A-D and A-B).

Still another approach to calculating slope is to compute a mean elevation for the center of a square patch by averaging the elevations at the four corners of the patch. If the patch is square then axes connecting the centroid and the four corners are perpendicular and spatial derivatives can be calculated for each of these triangular patches. Thus, for the example above the average elevation would be calculated as

$$AVEL = (A + B + C + D) / 4$$

and slope would then be calculated using the differences between the average elevation at the centroid and the elevations at each corner (call this the 3C, or 3 point around the Centroid case).

$$Slope^{O} \ (3C\ case) = ArcTan\sqrt{\left(\frac{A-AVEL}{15\sqrt{2}}\right)^2 + \left(\frac{AVEL-B}{15\sqrt{2}}\right)^2} \tag{7}$$

Reasonableness of the Surface Partitions

Such subdivisions of a DEM grid might be interpreted solely as an academic exercise. Indeed, it is an academic exercise and it illustrates much about the way slope might be calculated and interpreted. But, it is more, for there are situations where there is utility in making such subdivisions. Horn (1981) argued for the weighted averages in certain situations. He also noted that one could opt for less smoothing and suggested the models I referred to as the 4PT and 3E cases. Douglas (1986) employed the 3C case to calculate slope to resolve conflicts in threading contours through a gridded DEM.

The ability to calculate slope with any of these methods presupposes that the DEM is free of any blunders or significant errors (Carter, 1989a. If there are significant errors, each of these methods of slope calculation will give erroneous results for none of methods provide sufficient smoothing to eliminate the effects of a significant error. Since all USGS gridded DEMs report elevations to the nearest whole number, the precision with which the data are reported will have an impact on the calculation of slope. If it could be assumed that the elevations were accurate to the nearest whole digit, then elevations reported in feet would permit finer resolution of detail than elevations reported to the nearest meter. This would be particularly important when trying to resolve slope to define watershed boundaries or fairly shallow stream courses.

Statistical and Graphical Effects

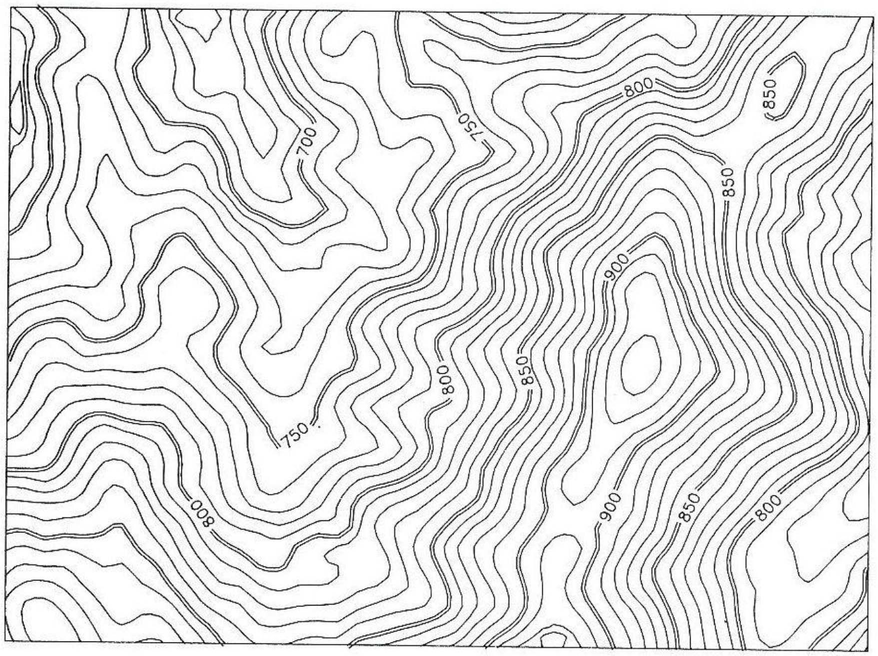
To show the effects of the spatial resolution in the calculation of the spatial derivative, descriptive statistics were computed on two sample datasets (Table 1) and three plots of slope and aspect were generated for the smaller dataset (Figures 2-5). Both of these datasets were extracted from the USGS 7 1/2 minute Thunderhead Mountain, North Carolina and Tennessee DEM generated on the GPM-II (USGS, 1987). The datasets were tested to verify that they were free of local errors by plotting contours and calculating DEM twist indices (Carter, 1989b).

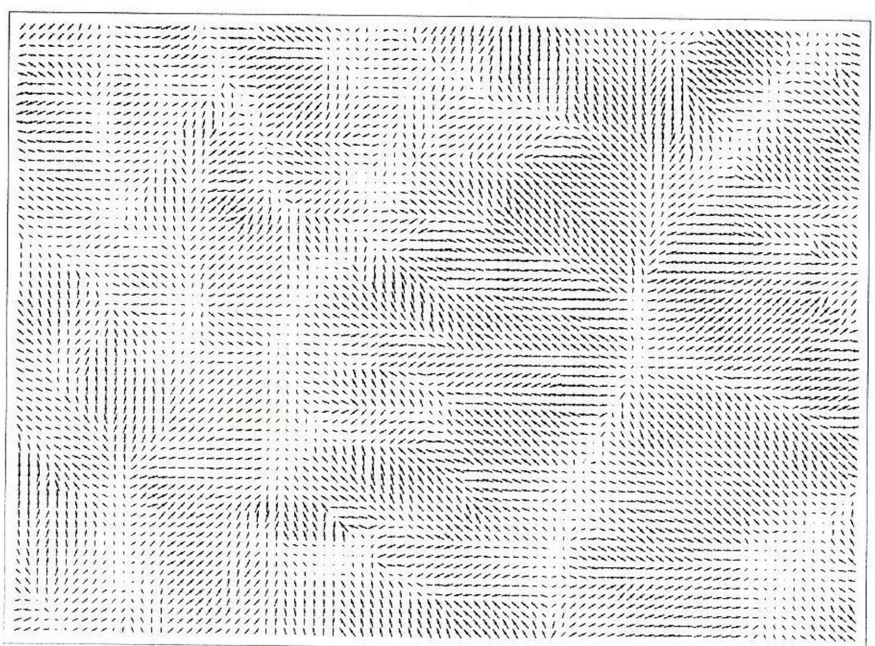
TABLE 1

	Degrees of Slope						
	Min	**Max**	**Mean**	**Median**	**SD**	**Skew**	**Kurt**
33 Row by 45 Column Dataset along the north slope of the Blue Ridge, Great Smoky Mountains National Park							
9PT	1.7	36.7	20.8	21.3	7.1	-0.33	-0.56
5PT	1.0	38.4	21.1	21.7	7.3	-0.36	-0.42
4PT	1.4	39.5	21.7	22.3	7.3	-0.39	-0.33
3Center	0.0	41.4	21.9	22.5	7.3	-0.35	-0.32
3Edge	0.0	42.0	22.0	22.4	7.3	-0.32	-0.27
74 Row by 107 Column Dataset along the main ridge of the Blue Ridge, Great Smoky Mountains National Park							
9PT	0.5	42.4	24.0	24.7	6.6	-0.40	0.01
5PT	1.0	43.2	24.6	25.2	6.7	-0.39	0.09
4PT	0.0	45.5	25.1	25.7	6.9	-0.34	0.11
3Center	0.0	48.2	25.3	25.9	7.1	-0.37	0.25
3Edge	0.0	51.1	25.4	25.9	7.3	-0.33	0.21

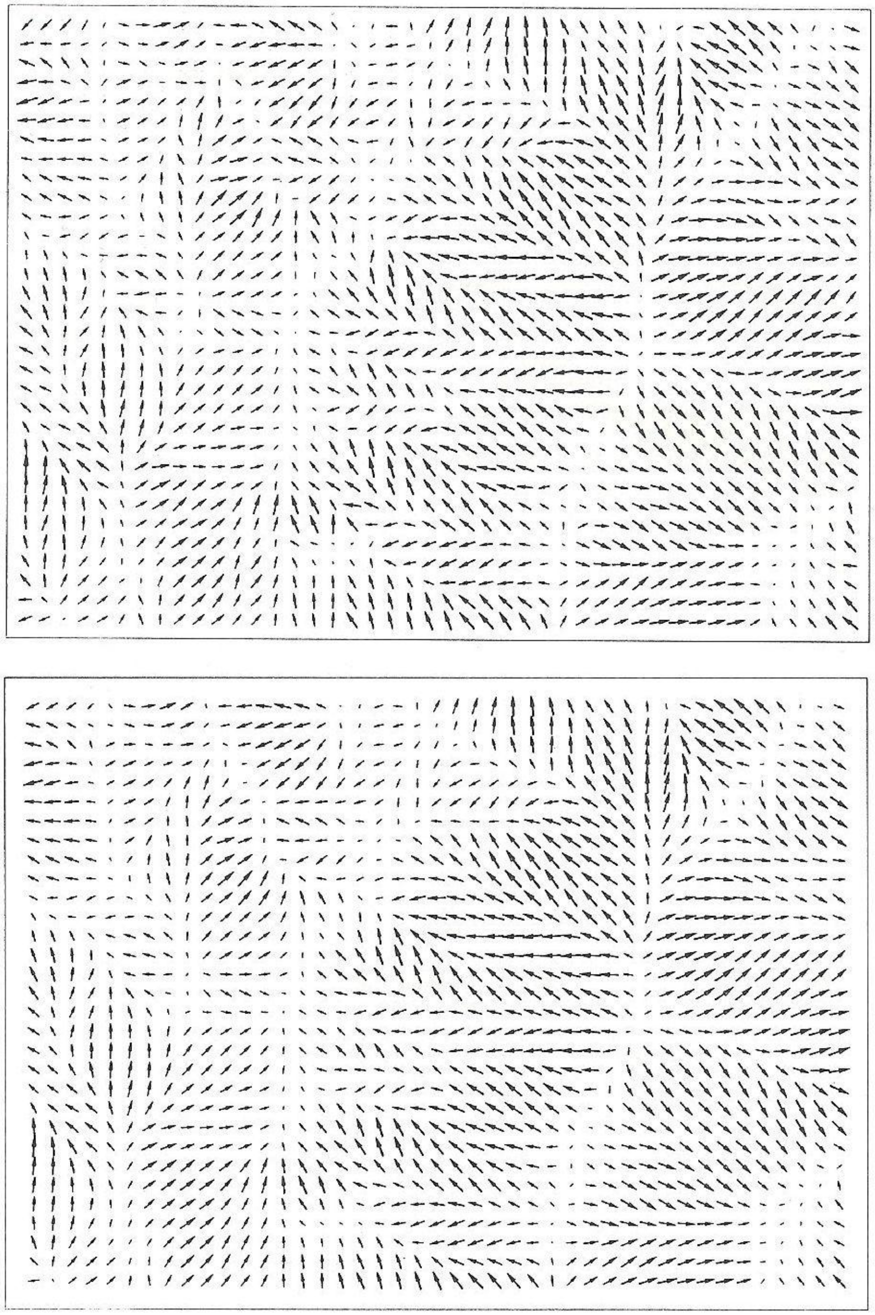
SD is standard deviation, Skew is skewness, and Kurt is kurtosis

The descriptive statistics were generated using the UNIVARIATE PROC in SAS (1985) for each of the five approaches to selecting elevations for use in the spatial derivative. These two datasets were found to be fairly normally distributed. Probably the most important thing to be noted is the effect of spatial resolution on the extremes of slope. In both cases, the 9PT method of point selection produced the smallest range of slope values and the lowest mean and median. This method also had the lowest measure of kurtosis, indicating that the





Figures 2 and 3 - Contour map and 3E arrow plot of the 33 Row by 45 Column dataset from the USGS Thunderhead Mountain 7 1/2' DEM. Contour map was made with Surface II and arrow plots from a program by the author.



Figures 4 and 5 - Slope and aspect for the 33 Row by 45 Column dataset. In the upper plot slope was calculated with the 4PT sampling and in the lower plot the 9PT sampling was used.

distribution produced by this method tends toward being platykurtic (flatter). At the other extreme, the 3E method of elevation point selection produced the greatest extremes, the largest means and medians, and distributions slightly more leptokurtic (peaked).

Plots with arrows proportional in length to the degree of slope illustrate the similarities and differences between three of the different methods of selecting elevations from gridded DEMs for calculating slope using the spatial derivative. No attempt was made to plot arrows for the 3C model -- that of the interpolated centroid value, because the centers of the unit areas do not form a regular grid. The 5PT model was not plotted because it would be intermediate between the 9PT model and the 4PT model and little would be gained by showing such an intermediate plot. The arrows in all of these plots are scaled so that the same degree of slope is represented by the same length of arrow in the 9PT and 4PT plots and the length of the arrows are 50% smaller in the 3E plot.

Based on this 33 Row by 45 Column matrix of elevations along the north flank of the Great Smoky Mountains National Park, mean slope was found to vary from 20.8 degrees to 22.0 degrees and the maximum slope varied from a low of 36.7 degrees to a high of 42.0 degrees, depending on how the elevations were selected to be used in the spatial derivative.

Little differences can be seen in the two plots of arrows (9PT and 4PT) based on the 30 meter spacing between centers of arrows. What differences one might detect could be accounted for by the fact that the center of the arrows in one plot are offset by 15 meters in both directions from the other plot. The plot of arrows based on a 15 meter spacing between arrows (3E) does not display the 'flowing' surface of the other plots. In many cases there seem to be blocks of arrows all trending in the same direction. At the edges of these blocks changes look like disruptions in some cases. At one place in Figure 3, southwest of the center of the plot, there appears to be an irregularity. This did not show up in the other two plots. A study of the elevations at this site suggests that one elevation value is off by perhaps 4 - 8 meters. If this were the case the 3E plot flaunts the problem while the 4PT and 9PT plots smooth it out. If, on the other hand, these elevations are correct, then the 3E plot reveals the texture of the surface while the 4PT and 9PT plots obscure the finer details of the surface.

The value of this work is in showing the trends in the differences between the different methods of selecting the elevations for the calculation of slope. If one is concerned with extremes and is likely to report such, then there is an obligation to report how slope was calculated and what spatial resolution was used to select elevations. Calculating slope from gridded DEMs is more complex than simply computing 'rise over run.'

Conclusions

The spatial derivative is probably the most common way to calculate slope from gridded DEMs. Depending on the quality of the data base and what the analyst is looking for in the data, a variety of spatial resolutions may be employed to use the spatial derivative. In the models used for this study, only subtle differences were produced using five different methods to sample elevations to compute the spatial derivative.

The descriptive statistics of the datasets of slope showed that as fewer elevations were incorporated in the calculation of slope and as the spatial resolution of the sample points became finer, the maximum slope became larger, the range of slope values became greater, and the distribution tended to become more leptokurtic (peaked). This should not be surprising for by going the other direction and using more points and/or a broader spatial resolution one should expect to get a smoothing effect.

Small differences in the effects of spatial resolution were seen in plots of arrows showing slope steepness and direction. The plot based on the broader spatial resolution seemed to show smoother flow patterns and a more coherent trend. The plot based on using only three elevations per slope calculation in the finest resolution possible showed many parallel vectors which tended to give less coherence to the overall surface.

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